Exercises supplementing those in James Stewart's *Calculus, Early Transcendentals*, 8th Edition

This is a collection of exercises that I have put on homework sheets over the years in Math (H)1A, (H)1B, and 53, supplementing the exercises in the text. Some I have assigned for the students to do and hand in; others I have suggested for students ''interested in further interesting and/or more challenging problems'' to try on their own. (Under the latter heading I have also recommended some of Stewart's exercises, and many of his ''Problems Plus''.) How many of my students work on problems so recommended, I don't know. (In H1B, at least, I know that some do, since they have asked questions about them.)

This collection ranges from exercises that are comparable in difficulty to those in the text, but fill some gap or give some interesting perspective on the material, through exercises that would be ''challenge problems'' in a regular courses and good exercises for an honors courses, to some that could be ''challenge problems'' for an honors course. I leave it to you to distinguish which are which.

I follow Stewart's notation and terminology, except that I write sequences as (a_n) rather than $\{a_n\}$.

I have given the exercises numbers indicating the relevant section of Stewart, and continuing where the numbering of Stewart's exercises in that section leave off, except, occasionally, when I have named one of my exercises as an ''additional part'' of one of Stewart's.

Most of these exercises were composed when I was teaching from earlier editions. I have tried to match them with the correct sections of the current edition, update references to the text, and drop exercises which are similar to ones which Stewart has added, or which do not fit any material in the current edition. I would appreciate having errors in these adjustments that anyone notices pointed out, as well as any other comments or suggestions on this packet.

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Principles of Problem Solving (p. 76), ''Exercise 20(a′)''. Same question as in 20(a), but with $f_{n+1} = f_n \circ f_0$ instead of $f_0 \circ f_n$.

§2.4, ''Exercise 44(d)''. Suppose $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ are both ∞. Show that

$$\lim_{x\to a} (f(x) + g(x)) = \infty.$$

§2.4, ''Exercise 45''. Suppose f is a function on the real line, and a and L are real numbers. Prove that $\lim_{x\to a} f(x) = L$ holds if and only if the following holds:

(∗) For every open interval (s, t) containing L there exists an open interval (u, v) containing a such that for every real number $x \neq a$ in (u, v), the real number $f(x)$ lies in (s, t).

This gives another formulation of the concept of limit. (If you are unsure of the meaning of ''open interval'', see the topic ''Intervals'' on pp. A3-A4.)

To prove the above result, you must show two things: That if $\lim_{x\to a} f(x) = L$ holds (under the book's definition), then condition (∗) holds, and conversely, that if condition (∗) holds, then $\lim_{x\to a} f(x) = L$ holds.

§2.4, ''Exercise 46''. Let f be the function defined by the conditions that if x is irrational, then $f(x) = 0$, while if x is a rational number whose expression in lowest terms is m/n (i.e., such that $x = m/n$ where m and n have no common divisor, and n is positive), then $f(x) = 1/n$. Show that for every real number a (rational or irrational), $\lim_{x\to a} f(x) = 0$.

§2.4, ''Exercise 47''. Exercises 44(a) in Stewart, and 44(d) above, show us two sorts of function $g(x)$ having the property that whenever one adds them to a function $f(x)$ satisfying $\lim_{x\to a} f(x) = \infty$, one must get a sum whose limit as $x \to a$ is again ∞; namely, functions $g(x)$ that approach a (real) limit, and functions $g(x)$ that approach ∞ as $x \to a$.

(a) Find a characterization of *all* functions $g(x)$ having this property. I.e., find an easily described condition on a function $g(x)$ such that if $g(x)$ satisfies your condition, then for every function $f(x)$ which satisfies $\lim_{x\to a} f(x) = \infty$ one also has $\lim_{x\to a} (f(x) + g(x)) = \infty$, and such that, conversely, if for every function $f(x)$ which satisfies $\lim_{x\to a} f(x) = \infty$ one has $\lim_{x\to a} (f(x) + g(x)) = \infty$, then $g(x)$ satisfies your condition,

If you don't know where to start, you might look at the function $g(x) = \sin 1/x$, and try to decide whether it has the desired property for $a = 0$.

(b) One can also ask: Which functions $g(x)$ have the property that for *some* (rather than *every*) function $f(x)$ which approaches ∞ as $x \to a$, one has $\lim_{x\to a} (f(x) + g(x)) = \infty$? Can you answer this question, or at least show by example that the answer is not the same as the answer to part (a)?

§2.4, "Exercise 48". Show by examples that in the situation of Stewart's §2.4, Exercise 44, if $c = 0$, then (for various choices of f and g), $\lim_{x\to a} (f(x)g(x))$ can be 0, 1, -1, ∞, $-\infty$, or undefined (i.e., neither any real number nor ∞, nor $-\infty$).

§2.4, "Exercise 49". Given real numbers a and b, consider the behavior of $ax^{-1} + bx^{-2}$ as x approaches 0^+ (i.e., approaches 0 from above). Under what conditions on a and b will this function have limit $+\infty$? $-\infty$? Are there cases in which it has neither of these limits? If so, what does it do in such cases? Can you prove your assertions?

§2.5, "Exercise 74". (a) Prove: If f is an *increasing* function (i.e., a function satisfying $x < y \Rightarrow f(x) < f(y)$), whose domain is an open interval (a, b), and whose image (the set of values it takes on) is an open interval (c, d), then f is continuous.

(b) Deduce from (a) that the same result is true for a *decreasing* function. (You will get full credit for this part, even if you do not do part (a), as long as you correctly show that this result follows from the statement of (a).)

§2.5, "Exercise 75". (a) Show (using the precise definition of limit) that if f is a real-valued function on an interval (a, b) whose values are everywhere positive, then $\lim_{x\to a^+} f(x) = 0$ if and only if $\lim_{x\to a^+} f(x)^{-1} = \infty$. (Here $f(x)^{-1}$ means $1/f(x)$. When one means the inverse function to f, one writes $f^{-1}(x)$.)

(b) Deduce that if f is a *continuous* real-valued function on an interval (a, b) whose values are nowhere zero, then $\lim_{x\to a^+} f(x) = 0$ if and only if either $\lim_{x\to a^+} f(x)^{-1} = \infty$ or $\lim_{x\to a^+} f(x)^{-1} = -\infty$.

§2.5, "Exercise 76". Suppose f and g are continuous at a. Show that the function h defined by

$$h(x) = \max(f(x), g(x))$$

is also continuous at a. (Here $\max(x, y)$ means the maximum of x and y, i.e., whichever of them is larger if they are not equal, or their common value if they are equal.)

§2.6, "Exercise 82". Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is an increasing function (i.e., has the property that whenever $a < b$ we have $f(a) < f(b)$), and that there exists a set S of real numbers such that $\{f(s) \mid s \in S\}$ is unbounded above (i.e., for every real number N there is some s such that $f(s) > N$).

Show that $\lim_{x\to\infty} f(x) = \infty$.

§2.6, "Exercise 83". Suppose $\lim_{x\to a} g(x) = \infty$, and $\lim_{x\to\infty} f(x) = L$. Prove that $\lim_{x\to a} f(g(x)) = L$.

§2.6, "Exercise 84". Suppose f is an increasing function whose domain is an interval (a, ∞), where a is a real number, and whose range is the whole real line, $(-\infty, \infty)$. Show that $\lim_{x \to a^+} f(x) = -\infty$, and $\lim_{x \to \infty} f(x) = \infty$.

§3.2, "Exercise 65". Suppose p and q are polynomials, and $m < n$ are positive integers. Prove that the mth derivative of $p(x)q(x)^n$ is divisible by $q(x)^{n-m}$. (Suggestion: Use Mathematical Induction, reviewed on pp. 72 and 74.)

§3.2, "Exercise 66". Suppose p and q are polynomials, and n a positive integer. Prove that the nth derivative of $p(x)/q(x)$ can be written as a rational function with denominator $q(x)^{n+1}$; i.e., that there is a polynomial $a_n(x)$ such that $D^n(p(x)/q(x)) = a_n(x)/q(x)^{n+1}$. (Suggestion: Use Mathematical Induction, reviewed on pp. 72 and 74.) Note that you are not asked to find a formula for $a_n(x)$; that would be much more difficult.)

§3.3, "Exercise 59". The point of this exercise is to make precise the reasoning of Example 4, p. 195.

Suppose f is a function, and p (for "period") is a positive integer such that f is p-times differentiable, and such that $f^{(p)} = f$. Show by Mathematical Induction (pp. 72, 74) that the function f is in fact n-times differentiable for every positive integer n, and that each of its higher derivatives $f^{(n)}$ equals one of the p functions $f, f', f'', \ldots, f^{(p-1)}$.

§3.5, "Exercise 81". (a) Let f be a differentiable function, and consider the curve $x = f(y)$. Suppose a differentiable function g is defined implicitly by that equation; i.e., that the curve $y = g(x)$ lies in the curve $x = f(y)$. Obtain a formula for g' by implicit differentiation.

(b) Apply the above result to the function $f(t) = t^n$ for n a positive integer: Name a function g that is defined implicitly by $x = f(y)$, and assuming this function is differentiable, get a formula for its derivative using the result of (a). Check your formula against what you know.

(c) Likewise, for $f(t) = e^t$, name a function defined implicitly by $x = f(y)$ and assuming it is differentiable, apply the result of (a) to find its derivative.

§3.11, "Exercise 60". This exercise is like Stewart's Exercise 59, but more is left to you to discover.

Show that almost every function of the form $(a\,e^{rx} + b)/(c\,e^{rx} + d)$, where a, b, c, d, r are real constants, has the same graph as some hyperbolic function, but shifted and stretched in appropriate ways. (You need to find the appropriate hyperbolic function(s), the kinds of stretching and shifting occurring, and the cases that must be excluded for your result to hold, just as $a=0$ and $b=0$ were excluded in Exercise 59.)

§4.3, "Exercise 94". Let f be a continuous function on an interval I.

(a) If f is differentiable, show that the following conditions are equivalent.

(i) f is concave upward in the sense of the Definition on p. 296.

(ii) For all $x_0 < x_1$ in I, the part of the curve $f(x)$ with $x_0 < x < x_1$ lies below the line segment connecting $(x_0, f(x_0))$ and $(x_1, f(x_1))$.

(b) Give an example of a continuous function f which is not differentiable everywhere on its interval of definition (so that condition (i) is meaningless, since with no derivative, one can't define the tangent line), but which satisfies condition (ii) above.

Remark: (ii) is a more common way of defining concavity than Stewart's. Another definition, in the spirit of (i), but meaningful and equivalent to (ii) whether or not f is differentiable, is (i′) For every $x_0 \in I$ there exists a line passing through $(x_0, f(x_0))$ and lying below the curve $y = f(x)$.

All the above definitions are a little ambiguous, due to a slight ambiguity in the words ‘‘above’’ and ‘‘below’’. If we take them to mean $\geq$ and $\leq$, then we get a standard version of the concept of concavity. If we take them to mean ‘‘$>$ and $<$, except at the obvious points where equality must hold’’ (namely, the point of tangency in Stewart’s definition, the points x_1 and x_2 in condition (ii) above, etc.), then we get a concept called ‘‘strict concavity’’. For instance, the function $f(x) = |x|$ is concave upward, but not strictly concave upward.

§4.3, ‘‘Exercise 95’’. Suppose f is a differentiable function on an interval (a, b). We have seen in Stewart that if f' is positive on the whole interval, then f is increasing. However, though the function x^3 on the interval $(-1, 1)$ is everywhere increasing, it satisfies $f'(0) = 0$, hence its derivative is *not* everywhere positive. This exercise notes some criteria that apply to such cases. Parts (a), (b), (c), (d) are successively more difficult.

(a) Suppose f' is positive at *all but one point* of (a, b). Show that f is increasing on the *whole* interval.

(b) Suppose f' is positive at all but a *finite number* of points of (a, b). Again show that f is increasing on the whole interval.

(c) Let g be the function on $(-1, 1)$ given by $g(x) = (x \sin x^{-1})^2$ if $x \neq 0$, and $g(0) = 0$. (Thus, g is 0 at infinitely many points, and positive wherever it is not 0.) Show that if f is a function satisfying $f' = g$, then f is increasing on the whole interval.

(d) Find and prove a condition on f' which is *necessary and sufficient* for f to be increasing. (Thus, (a)-(c) will be special cases of your result.)

§4.5, ‘‘Exercise 77’’. Show that the curve $y = \sqrt{(x^2 + 1)}$ has asymptotes $y = x$ and $y = -x$.

§4.5, ‘‘Exercise 78’’. Exercises 72, 73, and ‘‘77’’ above give some examples of slant asymptotes of hyperbolae. Let us generalize these, and note a related result.

Suppose a, b, c are real numbers, with $a > 0$.

(a) Find the asymptotes of the curve $y = \sqrt{(ax^2 + bx + c)}$.

(b) Show that $y = -\sqrt{(ax^2 + bx + c)}$ has the same asymptotes as $y = \sqrt{(ax^2 + bx + c)}$.

§4.5, ‘‘Exercise 79’’. (a) Show that the curve $y = \sqrt[3]{(x^3 + 1)}$ has the asymptote $y = x$.

(This requires more original thought than the preceding two exercises, since we haven’t studied methods of finding limits of differences involving cube roots. Suggestion: Either figure out how to ‘‘rationalize numerators’’ of such expressions, or use l’Hospital’s Rule.)

(b) Similarly determine the asymptotes of the curves $y = \sqrt[3]{(x^3 + x)}$, $y = \sqrt[3]{(x^3 + x^2)}$, and generally, $y = \sqrt[3]{(ax^3 + bx^2 + cx + d)}$.

§4.5, ‘‘Exercise 80’’. (a) Find the regions of increase, decrease, and upward and downward concavity of the curve $y = \sqrt[3]{(x^3 + 1)}$ (part (a) of the preceding exercise), and sketch the curve.

(b) The above curve has a kind of symmetry not discussed by Stewart. Express this symmetry in a precise way. (Note that symmetry of a curve about the y-axis can be expressed, ‘‘If (x, y) lies on the curve, so does $(-x, y)$’’, while symmetry about the origin says, ‘‘If (x, y) lies on the curve, so does $(-x, -y)$’’. You should get a similar description for the symmetry of the above curve.)

§4.9, "Exercise 80". (a) In this part we will see that functions which are "close" to one another have antiderivatives which are also "close".

Let f and g be functions on an interval (a, b), and let d be a positive real number, such that $|f(x) - g(x)| < d$ for all $x \in (a, b)$. Suppose that f has an antiderivative F, and that g also has an antiderivative. Show that g will in fact have an antiderivative G such that $|F(x) - G(x)| < d(b - a)$ holds for all $x \in (a, b)$.

(b) Is it true that in this situation *every* antiderivative G of g will satisfy the above inequality? Why or why not?

(c) In this last part, you will show that no analogous result is true for *derivatives*. Namely, show that for any interval (a, b) and any positive real numbers d and N, there exist differentiable functions f and g on (a, b) such that $|f(x) - g(x)| < d$ holds for all $x \in (a, b)$, but such that not all $x \in (a, b)$ satisfy $|f'(x) - g'(x)| < N$. (Suggestion: Let $g = 0$. Then, the problem is just one of finding f with appropriate properties.)

§4.9, "Exercise 81". (a) Find the most general antiderivative of the function $x^{-1/3}$.

(b) Find the most general antiderivative of $x^{-1/3}$ which can be made into a *continuous* function on the whole real line by giving it an appropriate value at $x = 0$.

Appendix E, "Exercise 51". (*As preparation for* §5.2.) Given a sequence of real numbers $a_1, a_2, \ldots, a_n, \ldots$, it may be hard to find a general formula for $\Sigma_{i=1}^{n} a_i$. However, given such a sequence, and a formula $\Sigma_{i=1}^{n} a_i = b_n$, I claim that it is easy to determine whether the formula is true. Namely,

(a) Given sequences of real numbers $a_1, a_2, \ldots, a_n, \ldots$ and $b_0, b_1, b_2, \ldots, b_n, \ldots$ with $b_0 = 0$, prove that the formula $\Sigma_{i=1}^{n} a_i = b_n$ holds for all positive integers n if and only if $b_n - b_{n-1} = a_n$ holds for all positive integers n.

(b) Using the above result, prove formula (e) on p. A37.

§5.3, "Exercise 87". This exercise concerns cases where the Fundamental Theorem of Calculus (part 1) does *not* apply. To prepare for these, you should do

(a) Suppose f and g are two functions defined on an interval $[a, b]$ which agree except at a single point $c \in [a, b]$. I.e., suppose $f(x) = g(x)$ for all $x \neq c$ in that interval, while $f(c)$ and $g(c)$ are defined but are not equal. Show that if the integral $\int_a^b f(x)\,dx$ is defined, then $\int_a^b g(x)\,dx$ is also defined, and is equal to $\int_a^b f(x)\,dx$.

Using the above result you should not find it hard to do

(b) Let $h(x)$ be the function on $\mathbb{R}$ which is 0 except at $x = 1$, and has $h(1) = 1$. Show that $\int_0^x h(x)\,dx$ is defined for all x, and is differentiable, but that its derivative is not everywhere equal to $h(x)$.

Similarly,

(c) Let $j(x)$ be the function which is defined to have value -1 when x is negative, $+1$ when x is positive, and 0 when $x = 0$. Show that $\int_0^x j(x)\,dx = |x|$ for all x, but that the conclusion of the Fundamental Theorem of Calculus (part 1) fails for this function when $x = 0$.

§5.5, "Exercise 95". Suppose f is a continuous function which is periodic, with period p, i.e., which satisfies

$$f(x+p) = f(x)$$

for all x. Show that the integral of f over a period is independent of the choice of starting-point; i.e., that for all real numbers a and b,

$$\int_a^{a+p} f(x)\,dx = \int_b^{b+p} f(x)\,dx.$$

(There are several possible ways to prove this.)

§6.1, "Exercise 62". Let a be a real number >1. We would like to find $\int_1^a \ln x\,dx$, but we haven't seen any antiderivative of $\ln x$. However,

(a) Describe and sketch a region whose area represents the desired integral.

(b) Find the area of this region by integrating with respect to y.

(c) Renaming the x in the above integral as "t", and the a as "x", get a formula for $\int_1^x \ln t\,dt$, valid for all $x > 1$.

(d) Show by differentiation that the function you found in (c) is an antiderivative of $\ln x$ not only for $x > 1$, but for all $x > 0$ (i.e., all x in the domain of $\ln x$).

§7.1, "Exercise 75". (a) Show that for every integer $n \geq 0$, one has $\int x^n e^x\,dx = p_n(x)\,e^x + C$ for some polynomial p_n of degree n.

(b) Writing the polynomial $p_n(x)$ of part (a) as $a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0$, expand the equation $(p_n(x)\,e^x)' = x^n e^x$, and solve to find $a_n, \ldots, a_0$ exactly. Thus, get an exact formula for $\int x^n e^x\,dx$.

(c) Deduce a formula for $\int e^{x^{1/n}}\,dx$ for n a positive integer. (Hint: The n in this part need not be the same as the n in the case of (b) which you will make use of.)

§7.1, "Exercise 76". Evaluate the integral $\int ((x-1)/x^2)\,e^x\,dx$. (Suggestion: Begin by writing the integrand as the sum of two functions, and applying integration by parts to one of them, leaving the other unchanged.)

§7.2, "Exercise 71". Evaluate the integral $\int \sin x\,\tan^2 x\,dx$.

§7.2, "Exercise 72". Evaluate the integral $\int \sec^3 x\,\csc x\,dx$. (This is not hard to solve using material in a later section; but see whether you can do it using the ideas of §7.2.)

§7.2, "Exercise 73". In Example 9 on p. 484, Stewart says that $\int \sin 4x \cos 5x\,dx$ "could be evaluated using integration by parts", but does it a different way. Show how to do this by integration by parts. (Hint: The pattern will be like that of Example 4 on p. 472.)

The result looks different from the answer he gets; show how these results can be reconciled.

§7.3, "Exercise 45". In a class some years ago, I gave as a problem the integral $\int (6x - x^2)^{-1/2}\,dx$. The answer one gets by completing the square and making a trigonometric substitution is $\sin^{-1}(x/3 - 1) + C$. However, one student showed me a different way she had approached the problem: she had written $(6x - x^2)^{-1/2}$ as $x^{-1/2}(6 - x)^{-1/2}$, noted that $x^{-1/2}\,dx = 2\,d(x^{1/2})$, and made the substitution $u = x^{1/2}$. This turned the integral into $2\int (6 - u^2)^{-1/2}\,du$, and trigonometric substitution applied to this integral leads to the answer $2\sin^{-1}(x/6)^{1/2} + C$.

One might suspect, therefore, that $\sin^{-1}(x/3 - 1) = 2\sin^{-1}(x/6)^{1/2}$, which in particular would say that $\sin(2\sin^{-1}(x/6)^{1/2}) = x/3 - 1$. But this is not so: if you apply the sine function to $2\sin^{-1}(x/6)^{1/2}$ and use the double angle formula, you get an algebraic expression in x, but it is not $x/3 - 1$. How can these answers be reconciled?

§7.4, "Exercise 76". In "§7.2, Exercise 72", I challenged you to attempt the integral $\int \sec^3 x\,\csc x\,dx$, mentioning that it could be done more easily using a method from a later reading. Do that integral now, by making the substitution $u = \sin x$, verifying that it converts the integrand into a rational function of u, integrating this by the methods of §7.4, and, of course, converting the result back to a function of x.

§7.7, "Exercise 51". In this exercise, you will work out the details of the proof of the Error Estimate for the Midpoint Rule.

We begin by considering the $n = 1$ case; i.e., a single undivided interval, which we will denote $[c-h, c+h]$. Let f be a function defined on that interval, twice differentiable, and satisfying $|f''(x)| \leq K$ for all $x \in [c-h, c+h]$. Our idea will be to show that the function f is "close to" the linear function $g(x)$ given by $g(x) = f(c) + f'(c)(x-c)$, whose integral is exactly given by the Midpoint Rule. To do this, let $e(x) = f(x) - g(x)$.

In some of the calculations below, you may find properties **7** and **8** of the integral, given on p. 387, useful.

(a) Show that at $x = c$, $e(x)$ has value and derivative both equal to 0, and that for all $x \in [c-h, c+h]$, $|e''| \leq K$.

(b) Deduce that for all $x \in [c-h, c+h]$, we have $|e'(x)| \leq K|x-c|$.

(c) Deduce that for all $x \in [c-h, c+h]$, we have $|e(x)| \leq K(x-c)^2/2$.

(d) Deduce in turn that $|\int_{c-h}^{c+h} e(x)\,dx| \leq K h^3/3$. Translate this into a statement about the value of $\int_{c-h}^{c+h} f(x)\,dx$.

(e) Deduce from this the Error Bound for the Midpoint Rule given on p. 518. (To do this, describe the n intervals into which the interval $[a, b]$ in the statement of that rule is broken up for the application of the Midpoint Rule, and apply the above result to each of these. In justifying your computation, you may find the Triangle Inequality on p. A8 useful.)

§7.7, "Exercise 52". In this exercise, we will prove the Error Bound for the Trapezoid Rule.

As in the preceding exercise, we begin by considering a twice differentiable function f on an interval $[c-h, c+h]$ such that $|f''(x)| \leq K$ on that interval, and will first get what is in effect the $n = 1$ case of the desired law, then deduce the general case by applying that case to each subinterval in a decomposition of $[a, b]$.

But our method of getting the $n = 1$ case will be quite different from the method used for the Midpoint Rule.

(a) Apply integration by parts to $\int_{c-h}^{c+h} f(x)\,dx$, taking $f(x) = u$. This leaves a bit of freedom in the choice of v: it is uniquely determined only up to an added constant. (I.e., given one such function v, any function of the form $v + a$, for a a constant, will also do.) Choose your v so that $v(c) = 0$.

(b) Now apply integration by parts again, to the integral on the right-hand side of the formula you got in part (a), with $u = f'$. This time, choose your v so that it is 0 at one endpoint of $[c-h, c+h]$. Verify that this also makes it zero at the other endpoint, and that this greatly simplifies the non-integral term in the result.

(c) Use the assumption $|f''(x)| \leq K$ to bound the integral in your formula, and show that the resulting equation is the $n = 1$ case of the Error Bound for the Trapezoid Rule.

(d) Deduce the general case of the Error Bound for the Trapezoid Rule by applying the above result to each of the n subintervals into which that procedure subdivides $[a, b]$, and summing the results.

§7.7, "Exercise 53". In this exercise, you will prove the Error Bound for Simpson's Rule.

For simplicity, let us perform our main calculations with $n = 2$ and a function f on the interval $[-1, 1]$. Then, at the end, we can use changes of variables to turn estimates for a function on that interval into estimates that work on each of the pairs of adjacent intervals into which an interval $[a, b]$ is divided in the general case of

Simpson's rule. So in parts (a)-(d) below −

(∗) Let f be a function on $[-1,1]$ which is four times continuously differentiable, and whose fourth derivative has absolute value everywhere $\leq$ a positive real number K.

In our first two steps, we will reduce to a still more special case:

(a) Let f_{even} be the function on $[-1,1]$ defined by $f_{\text{even}}(x) = (f(x)+f(-x))/2$. Show that the error in the $n=2$ Simpson's rule approximation of $\int_{-1}^{1} f_{\text{even}}(x)\,dx$ is the same as the error in the $n=2$ Simpson's rule approximation of $\int_{-1}^{1} f(x)\,dx$, and that f_{even}, like f, has fourth derivative everywhere bounded above by K. Deduce that the error bound will be correct for all functions f satisfying (∗) if it can be shown correct for all such functions which are even.

(b) Given an even function f satisfying (∗), and any constant c, consider the function f_c defined by $f_c(x) = f(x) - cx^2$. Show that the error in the $n=2$ Simpson's rule approximation of $\int_{-1}^{1} f_c(x)\,dx$ is the same as the error in the $n=2$ Simpson's rule approximation of $\int_{-1}^{1} f(x)\,dx$, and that f_c, like f, has fourth derivative everywhere bounded above by K. By appropriate choice of c, conclude that the error bound is correct for all even functions f satisfying (∗) if it is correct for all such even functions satisfying $f''(0)=0$.

(c) Show that every *even* function satisfying (∗) satisfies $f'(0)=0$ and $f'''(0)=0$.

(d) Now suppose that f is, as above, an even function on $[-1,1]$ that satisfies $f''(0)=0$. Translate the statement of the Simpson's rule approximation of $\int_{-1}^{1} f(x)\,dx$ to a statement of an approximation of $\int_{0}^{1} f(x)\,dx$ in terms of its values at the endpoints 0 and 1, and translate the error bound that you wish to prove into a bound on the error that approximation.

In the next three parts, you will prove the correctness of the asserted bound for any four times continuously differentiable function f on $[0,1]$ whose fourth derivative is everywhere bounded by K, and which satisfies $f'(0)=f''(0)=f'''(0)=0$.

(e) Given f as above, apply integration by parts four times, using as your successive functions u the functions $f,\ f',\ f'',\ f'''$. As before, the functions v that you use are determined up to constants, and the key to the calculation will be in the choice of those constants. At the first step, choose the constant so that the term $uv|_0^1$ arising in the integration by parts gives precisely the linear combination of $f(0)$ and $f(1)$ that you obtained in your translation of Simpson's rule in part (d) above. In the remaining three steps, on the other hand, choose v so that $v(1)=0$. Simplify the resulting formula for $\int_0^1 f(x)\,dx$.

(f) Show that the functions v occurring in the last two steps above never change sign on $[0,1]$ (i.e., that each of them is either always ≤ 0 or always ≥ 0. You will only need this fact for the function occurring at the last step; but it is easiest to prove it for the next-to-last function, and deduce using this that it is true for the last one.)

(g) Deduce that the final integral that you get in the above calculations is bounded in absolute value by $K\,|\int_0^1 v(x)\,dx\,|$, where v denotes the function occurring in that role in the last of the above integrations by parts. Evaluate this integral.

(h) Going back through steps (d), (b) and (a), deduce the correctness of the error bound in the $n=2$ Simpson's rule approximation for any four times continuously differentiable function f on $[-1,1]$.

(i) Given any four times continuously differentiable function f on any interval $[a,b]$, and any even integer n, prove the error bound for the n-step Simpson's rule approximation for $\int_a^b f(x)\,dx$ by breaking the interval $[a,b]$ into n subintervals, grouping them into pairs of successive subintervals, making a linear change of variables on each of those pairs so that the integral there becomes an integral on the interval $[-1, 1]$, applying the result of (h) to that integral, and summing.

§7.7, "Exercise 54". (a) For d a positive integer, compute $\int_0^1 x^d dx$. Also compute the Simpson's Rule approximation of this integral with $n = 2$, and the error bound on that approximation, using the least possible K.

(b) Subtract the integrals from the approximations of part (a) for $d = 1, 2, 3, 4, 5$, and compare the resulting values with the error bounds you have obtained. (You could also consider the case $d = 0$, but the Simpson's Rule approximation will not be given in this case by the same formula as in part (a).)

§7.7, "Exercise 55". In the approximate integration rules we have studied, Δx denotes $(b-a)/n$. The three error bounds given on pp. 518 and 522 are all expressed in terms of $b-a$ and n. Write down for each of them an expression in terms of $b-a$ and Δx (but not n), and an expression in terms of Δx and n (but not $b-a$).

§7.8, "Exercise 83". Use the result of "§7.1, Exercise 75(b)" above to calculate $\int_{-\infty}^0 x^n e^x dx$.

§7.8, "Exercise 84". (a) Determine whether $\int_0^\infty \sin x^2\, dx$ converges. (Suggestion: use integration by parts: since $d\cos x^2$ contains a factor $\sin x^2\, dx$, let $v = \cos x^2$, let u be what it has to be, and note the behavior of the expression you get. You will need to apply the Comparison Test to the result, but that test is only stated in our text for functions satisfying $f(x) \geq g(x) \geq 0$. However, in fact, it is valid whenever $f(x) \geq |g(x)|$, which is what you need.)

(b) Can you find a continuous function f on $[0, \infty]$ such that f is unbounded (i.e., takes on values with arbitrarily large absolute value) but $\int_0^\infty f(x)\, dx$ is convergent?

§7.8, "Exercise 85". Suppose a is a real number, $f(x)$ is a continuous function defined for all $x \geq a$, which approaches some finite limit L as $x \to \infty$, and c is any positive real number.

(a) Show that $\int_a^\infty (f(x+c) - f(x))\, dx$ converges.

(b) Deduce that if $\int_a^\infty (f(x+c) + f(x))\, dx$ converges, then $\int_a^\infty f(x)\, dx$ converges.

(c) Deduce that $\int_\pi^\infty x^{-1} \sin x\, dx$ converges.

§7.8, "Exercise 86". The following examples illustrate how change of variables can convert improper integrals to ordinary integrals, or one sort of improper integral into another.

(a) The following improper integrals all converge. Work out the effects of the substitution $x = u^{-1}$ on each of them: $\int_1^\infty x^{-3} dx$, $\int_1^\infty x^{-2} dx$, $\int_1^\infty x^{-3/2} dx$.

(b) Say why $\int_0^{\pi^2} x^{-½} \cos x^{½}\, dx$ is an improper integral. Evaluate that integral using the substitution $u = x^{½}$, which converts it to an ordinary integral.

In all these cases, indicate why the original improper integral is equal to the new integral that you get by the substitution (taking as known that integration by substitution is valid for ordinary integrals).

§7.8, "Exercise 87". (a) For which pairs of real numbers a and b is the integral $\int_0^\infty (x^a + x^b)^{-1} dx$ convergent? (Prove your answer, using results from the reading.)

Once you have solved this problem, you should not find it hard to answer:

(b) For which finite families of real numbers $a_1, \ldots, a_n$ is the integral $\int_0^\infty (x^{a_1} + \ldots + x^{a_n})^{-1} dx$ convergent?

§8.1, "Exercise 47". Suppose $f(x)$ is an *increasing* continuously differentiable function on an interval $[x_0, x_0{+}\Delta x]$; and let us write $f(x_0) = y_0$ and $f(x_0{+}\Delta x) = y_0{+}\Delta y$. (There is no assumption that Δx or Δy is small; I am merely using these as convenient symbols for the changes in the values of x and of y between the endpoints.)

Let L denote the length of the curve $y = f(x)$ between the points (x_0, y_0) and $(x_0+\Delta x, y_0+\Delta y)$; i.e., $L = \int_{x_0}^{x_0+\Delta x} (1 + f'(x)^2)^{½}\, dx$. Prove using properties of the integral (without relying on geometric facts) that

$$((\Delta x)^2 + (\Delta y)^2)^{½} \leq L \leq |\Delta x| + |\Delta y|.$$

§8.2, "Exercise 40". Suppose one takes a point (x_0, y_0) in the plane, rotates it around the x-axis, getting a circle in a plane parallel to the y-z-plane, and then rotates this circle about the y-axis, getting a surface in 3-dimensional space. (As in §8.2, we are rotating a curve about an axis; but since it is not a curve in the x-y-plane, the picture is not quite as in that section.)

(a) Describe this surface (by equations and other conditions).

(b) Determine the area of the above surface (as a function of x_0 and y_0).

(c) Suppose we started with the same point (x_0, y_0), but first rotated it about the y-axis, then about the x-axis. Is the surface we get the same as the one we got by doing the rotation in the original order?

§8.2, "Exercise 41". How should formulas [4]-[8] on pp. 552-553 be modified to cover the case of a function f which is not assumed everywhere positive? (I.e., which may be positive for some values of x and zero or negative for others.)

§8.2, "Exercise 42". (a) Suppose f is an *even* differentiable function on the interval $[-a, a]$; i.e., satisfies $f(-x) = f(x)$. Show that for any $b \geq a$, the area of the surface of rotation of the curve $y = f(x)$ about the line $x = -b$ can be expressed in terms of b and the length of that curve.

(b) What can be said about the analogous situation where f is an *odd* function, i.e., satisfies $f(-x) = -f(x)$?

(The answers to both parts above are special cases of a result expressing the area of a surface of revolution of a curve in terms of the centroid of the curve; a variant of the Theorem of Pappus that Stewart states on p. 559.)

§8.3, "Exercise 52". Geometric intuition tells us that if we shift all points of a region in a given direction by a given amount, the centroid will be shifted in the same direction by the same amount.

(a) Check by calculation that in the formulas [9] on p. 564 for the centroid of the region between two curves $y = f(x)$ and $y = g(x)$, if we add the same constant c to both f and g (i.e., use the functions $f^*(x) = f(x) + c$ and $g^*(x) = g(x) + c$ in place of f and g), this will indeed increase the y-coordinate of the centroid by c.

This leads one to wonder whether, if one adds a constant c to f only, the y-coordinate of the centroid might increase by $½c$.

(b) Show that this not so. In fact, show by example that in some cases, adding a positive constant to f can decrease the y-coordinate of the centroid.

§8.5, "Exercise 22". Suppose μ, σ and k are real numbers with $\sigma > 0$. If a random variable X has normal distribution with mean μ and standard deviation σ (i.e., has probability density function given by formula [3] on p. 578), write down an integral expressing the probability (fraction of occurrences) of the variable having value at least k standard deviations above the mean; i.e., satisfying $X \geq \mu + k\sigma$.

Show that the value of this integral depends only on k, and not on μ and σ.

§8.5, "Exercise 23". Although $\int e^{-x^2} dx$ is not an elementary function, it can be approximated in various ways by elementary functions. Let us look for approximations of the ''tail'' of the distribution, $\int_x^{\infty} e^{-t^2} dt$, as $x \to \infty$.

(a) Show that for all x,

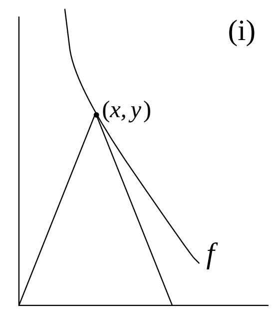
$$\int_x^{2x} e^{-(x^2+3x(t-x))} dt \quad < \quad \int_x^{\infty} e^{-t^2} dt \quad < \quad \int_x^{\infty} e^{-(x^2+2x(t-x))} dt .$$

(b) Evaluate the integrals on the right and left sides of the above inequality.

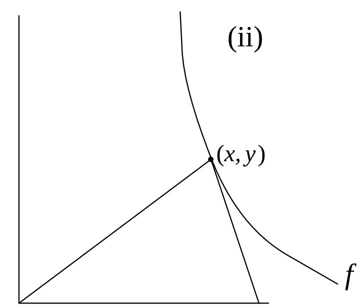
(c) Show that of the above two bounds on our integral, one has the property that the ratio of it to our integral approaches 1 as $x \to \infty$, while the other does not. (Hint: Use L'Hospital's rule.)

(d) Use the above bounds to get bounds on the probability discussed in ''Exercise 22''.

§9.3, ''Exercise 55''. Diagram (i) at right illustrates Stewart's Problem Plus number 15 on p. 638, which asks you to find all curves $y = f(x)$ with the property that if a line is drawn from the origin to any point (x, y) of the curve, and then a tangent is drawn to the curve at that point and extended to meet the x-axis, the result is an isosceles triangle, with equal sides meeting at (x, y). (*Note:* That sketch, and likewise sketches (ii) and (iii), which go with questions below, are *not* intended to correctly show the shapes of the curves, but simply to make clear the relation between the curve and the isosceles triangle in each case.) The statement in Stewart's Problem Plus that the triangle is isosceles can be translated into a condition on the slope of the curve at (x, y), giving a differential equation that can be solved by the methods you have learned.

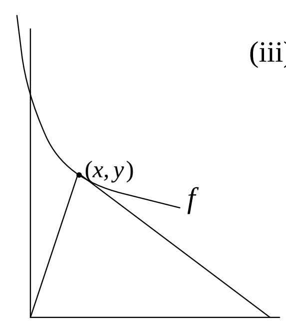


Varying that exercise, one can require, instead, that the triangle determined as above be isosceles with equal sides meeting at the origin (sketch (ii)), or at the point where the tangent hits the x-axis (sketch (iii)).



Each of these conditions can again be transformed into a differential equation, but the methods you have learned do not cover the equations one gets from (ii) and (iii). However, there is another method that can be used to solve all three equations:

(a) Show by geometric reasoning that if A is a curve with any of the three properties described, then the curve obtained by ''magnifying'' A by any nonzero real constant c; i.e., replacing each point (x, y) by (cx, cy), has that same property.



(b) Find a differential equation describing each of the above sets of curves. (Suggestion: In each case, for a point (x, y), determine the point(s) on the x-axis where the third vertex of an isosceles triangle of the indicated sort must lie. Compute the slope of the line from (x, y) to that point, and set that equal to the slope of the curve at (x, y).)

Now it is not hard to see that the fact proved in (a) implies that each of the equations you get in step (b) can be put in the form $y' = F(y/x)$. There is a change of variables that can be used in solving such equations: take $u = y/x$ and $v = \ln(x)$. Then the given differential equation will turn into an expression for dv/du in terms of u (not involving v). Hence by integrating, you can express v in terms of u. Transforming back into (x, y)-coordinates, you will get an equation for the desired curves.

(For a motivation for the change of variables $u = y/x$, $v = \ln(x)$, and also for the technique Stewart gives in §9.6 for solving homogeneous and nonhomogeneous first-order linear differential equations, see http://math.berkeley.edu/˜gbergman/ug.hndts/ode_symms.pdf. The approach to linear equations is developed in §§1-3 of that note; the approach to equations $y' = F(y/x)$ in §4.)

(c) Solve the equations from part (b) by the method indicated above.

(Remark: The calculations in cases (ii) and (iii) are more complicated than I would have expected from this elementary geometric problem; but the final answers, when put in geometric form, are simple and elegant.)

§9.5, "Exercise 36(b)". In §9.5, Exercises 35 and 36, the constant c represents the effect of air resistance. The value of that constant will in fact depend on the size and shape of the object.

Verify that if we take various objects having different masses, but whose mass and whose constant c are related in a certain way, then they *will* all fall at the same speed, contrary to what is stated at the end of Exercise 36.

Another possible assumption is that the objects are the same in shape and density, but differ in their scale. If we assume that for s the scaling factor, the mass of the object is proportional to s^3 while the constant c, arising from its surface area, is proportional to s^2, show that Stewart's conclusion that heavier objects fall faster again becomes correct.

§9.5, "Exercise 39". In §9.5, Exercise 23, you learned how to solve certain sorts of differential equations by making the substitution $u = y^{1-n}$, which reduces them to linear differential equations.

Describe, similarly, a class of equations that can be reduced to linear differential equations by the substitution $u = e^y$. (Suggestion: do that substitution in reverse, starting with a linear differential equation.)

§9.6, "Exercise 9(b)". We will call Exercise 9 as given in Stewart, p. 632 "part (a)". In part (b) below, we will get an idea of why the solutions to the Lotka-Volterra equations have the shapes shown on p. 629. (So in your answer to the final parts of that question, you cannot *assume* that they have those shapes.)

(b) In the solution of the separable differential equation given in part (a), an intermediate step is an equation of the form $F(W) = G(R) + c$. Sketch the functions $F(W)$ and $G(R)$ involved. Assuming a value of c chosen such that the above equation has solutions, describe how one could get such solutions from the graphs of F and G. Indicate why, for *almost* every value of W such that there exists at least one value of R satisfying that equation, there are in fact exactly two such values of R, and for almost every value of R such that there exists at least one value of W satisfying that equation, there are in fact exactly two such values of W. Explain why the set of values of R and the set of values of W that can occur in the solution are both bounded, and describe their maxima and minima in terms of features of your graphs.

§10.2, "Exercise 75". Let C be a curve which can be parametrized $x = f(t)$, $y = g(t)$, and let t_0 be a point in the common domain of f and g. Let P, Q and R be the values of y, dy/dx, and d^2y/dx^2 respectively at the point of C with $t = t_0$. Let a be a positive real number.

(a) Let C' be the curve obtained by "stretching" C by a factor of a, i.e., the curve parametrized by $x = af(t)$, $y = ag(t)$. Show that the values of y, dy/dx, and d^2y/dx^2 at the point of C' with $t = t_0$ are aP, Q, and R/a respectively.

(b) If S is the value of d^3y/dx^3 at the point of C with $t = t_0$, what would you guess is the value of d^3y/dx^3 at the corresponding point of C'? (You will not be graded on this part.)

(c) Let C'' be the curve with parametrization $x = f(at)$, $y = g(at)$. What are the values of y, dy/dx, and d^2y/dx^2 at the point of C'' with $t = t_0/a$?

(Note: Part (a) required calculation, and (c) can also be done in that way, but it can be done more easily by a quick bit of reasoning about the relation between the curves C and C''.)

§11.1, "Exercise 94". Prove the second of the two limit laws in the box on p. 697, namely

$$\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n\,.$$

Assume the two limits on the right exist.

§11.1, "Exercise 95". This exercise will lead you through a proof, based on the Axiom of Completeness of the real numbers, of the Extreme Value Theorem (p. 278), that is, the statement that every continuous function f on a closed interval assumes an absolute maximum and an absolute minimum.

Let f be a continuous function on a closed interval $[a,b]$. Let us define

$$S = \{x_0 \in [a,b] \mid \text{for every } x_1 \in [a,b] \text{ there exists an } x_2 \in [x_0, b] \text{ such that } f(x_2) \geq f(x_1)\}.$$

(Here $\{\ldots\}$ means "the set of all ...", and the vertical line $|$ means "such that". The idea is that S is the set of those x_0 for which we can hope that f has an absolute maximum at some $x \geq x_0$.)

(a) Show that the set S has a least upper bound c, and that $c \in [a,b]$.

(b) Show that for every $\delta > 0$, there is some $s \in S$ satisfying $|s - c| < \delta$.

We shall now show that $f(c)$ is an absolute maximum for the function f on $[a,b]$. To do this, suppose, by way of contradiction, that there were a $d \in [a,b]$ with $f(d) > f(c)$. Do the next few parts under this assumption.

(c) Show that there exists an $\delta > 0$ such that for all $x \in [a,b]$ with $|x - c| < \delta$, we have $f(x) < f(d)$.

(d) Deduce that every $t \in [a,b]$ satisfying $|t - c| < \delta$ belongs to S. (Idea: The values of $f(x)$ for x close to c are not big enough to affect whether a point belongs to S, so if one point in that range does, as shown by (b), then all do.)

(e) The above shows that $c \in S$; deduce from this that $c < b$.

(f) Show that (d) and (e) together contradict our choice of c. Since the above was deduced from the assumption that $f(c)$ was not an absolute maximum for f on $[a,b]$, conclude that $f(c)$ is indeed such a maximum.

(g) The above shows that every continuous function on a closed interval assumes an absolute maximum. Deduce from this that every continuous function on a closed interval also assumes an absolute minimum.

§11.1, "Exercise 96". (This exercise assumes you have done §11.1, Exercise 91, p. 706.)

The *harmonic mean* of two positive real numbers a and b is defined to be $2ab/(a+b)$, or, to put it in a form which shows the idea, $((a^{-1}+b^{-1})/2)^{-1}$. So, for instance, since the arithmetic mean of 2 and 4 is 3, the harmonic mean of $1/2$ and $1/4$ is $1/3$.

(a) Show that the analog of Exercise 91 is valid with "harmonic mean" in place of "geometric mean"; in other words, that by applying the operations of arithmetic mean and harmonic mean repeatedly, starting with a pair of positive real numbers, one gets sequences a_n and b_n which approach a common limit, which one might call the "arithmetic-harmonic mean" of a and b.

(b) Show that this "arithmetic-harmonic mean" is in fact the *geometric* mean of a and b.

§11.2, "Exercise 93". For all positive integers m and n, let $a_{m,n} = 1$ if $m = n$, $a_{m,n} = 0$ if $m \neq n$,

(a) For each positive integer m, evaluate $\Sigma_{n=1}^{\infty} a_{m,n}$. (Note that the summation is over n only, with m held constant. So you get a value for each m.)

(b) Using these values, compute $\lim_{m\to\infty} (\Sigma_{n=1}^{\infty} a_{m,n})$.

(c) On the other hand, for each positive integer n, compute $\lim_{m\to\infty} a_{m,n}$.

(d) Using these values, compute $\Sigma_{n=1}^{\infty} (\lim_{m\to\infty} a_{m,n})$.

(e) Are the results of (b) and (d) equal? I.e., do we have

$$\lim_{m\to\infty} (\Sigma_{n=1}^{\infty} a_{m,n}) = \Sigma_{n=1}^{\infty} (\lim_{m\to\infty} a_{m,n}) \ ?$$

(f) Modifying the above construction, find an example of a sequence of continuous functions $f_m(x)$ $(m \geq 1)$ on the real line which has the analogous property, with $\int_0^{\infty}$ replacing $\Sigma_{n=1}^{\infty}$. (Suggestion: use functions defined ''piecewise'' so that each is zero except on one interval $[n, n+1]$.)

(g) Can you get an example with the same property but for $\int_0^1$ rather than $\int_0^{\infty}$?

§11.3, ''Exercise 47''. This exercise extends Stewart's §11.3, Exercises 29 and 30 (p. 726).

(a) For what real numbers a and b does the series $\Sigma_{n=2}^{\infty} n^{-a} (\ln n)^{-b}$ converge?

(b) For what real numbers a, b and c does the series $\Sigma_{n=3}^{\infty} n^{-a} (\ln n)^{-b} (\ln \ln n)^{-c}$ converge?

§11.3, ''Exercise 48''. This exercise will show that the condition in the Integral Test that the function $f(x)$ be eventually decreasing is needed for the result to be true.

For any sequence $c = (c_1, c_2, \ldots)$ of numbers $c_n \in (0, ½)$, let f_c be the positive real-valued function on $[0, \infty)$ defined as follows:

For each positive integer n, and all x in the interval $[n - c_n, n + c_n]$, let $f(x) = 1 - (|n - x| / c_n)$.

For all points $x \in [0, \infty)$ not belonging to any of the above intervals, let $f(x) = 0$.

(a) Verify that for every such sequence c, the function f_c is continuous. (I don't ask here for an ε-δ-argument; just some observations that make the reasons intuitively clear.) For c the sequence $1/2, 1/3, 1/4, \ldots,$ sketch f_c on the interval $[0, 4]$.

(b) Show that for such sequences c, the integral $\int_0^{\infty} f_c(x)\, dx$ converges if and only if $\Sigma_1^{\infty} c_n$ converges.

(c) Show that for such sequences c, if we define $a_n = f_c(n)$, then the series $\Sigma_1^{\infty} a_n$ diverges, regardless of whether $\Sigma_1^{\infty} c_n$ converges.

But from (b) we know that for some sequences c, the integral of f_c converges. Hence by the result of (c), the convergence of that integral does not imply the convergence of the corresponding series $\Sigma_1^{\infty} a_n$.

§11.4, ''Exercise 47''. Give an example of a *convergent* series $\Sigma_1^{\infty} a_n$ with all $a_n \geq 0$ such that $\lim_{n\to\infty} n\, a_n$ is undefined, and an example of a *divergent* sequence with this property. (This contrasts with the result of Exercise 43.)

§11.6, ''Exercise 54''. Let (a_n) be a sequence.

(a) Suppose there is a nonnegative constant $c < 1$ and an integer N such that for all $n > N$ we have $\sqrt[n]{|a_n|} < c$. (I.e., suppose the sequence $(\sqrt[n]{|a_n|})$ is bounded above by some $c < 1$.) Show that Σa_n converges.

(b) Show that if the convergence of Σa_n follows from the root test, then it also follows from the above criterion.

(c) Give an example of a series which converges by the above criterion, but to which the root test is not applicable.

§11.6, "Exercise 55". let $\Sigma\, a_n$ be a series, and let the terms a_n^+, a_n^- be defined as in §11.6, Exercise 51. Show that the following conditions are equivalent:

(a) Some rearrangement of $\Sigma\, a_n$ is conditionally convergent.

(b) $\lim_{n\to\infty} a_n = 0$, but $\Sigma\, a_n^+$ and $\Sigma\, a_n^-$ both diverge.

§11.8, "Exercise 43". Suppose the power series $a_n x^n$, $b_n x^n$, and $(a_n + b_n)x^n$ have radii of convergence R_1, R_2 and R_3 respectively.

(a) Show that if $R_1 < R_2$, then $R_3 = R_1$.

(b) Deduce from part (a) that (without the assumption that $R_1 < R_2$) at least two of R_1, R_2 and R_3 are always equal. (Hint: Each of the three series can be obtained from the other two by term-by-term addition or subtraction.) If these radii are not all equal, what can one say about the relation between the two that are equal, and the third?

§11.8, "Exercise 44". If $a_n x^n$ is a power series with radius of convergence R, and k is a positive integer, find the radii of convergence of the power series $a_n^k x^n$, $a_n x^{kn}$, and $a_n^k x^{kn}$.

§11.8, "Exercise 45". Given a power series $a_n x^n$, express its radius of convergence in terms of the radii of convergence of the two power series $a_{2n} x^n$ and $a_{2n+1} x^n$.

§11.8, "Exercise 46". A function $f(x)$ is defined by

$$f(x) = 1 + 2x + x^2 + (2x)^3 + x^4 + \ldots .$$

That is, the n-th term is x^n if n is even, and $(2x)^n$ if n is odd. (To make this fit the definition of a power series, we can rewrite $(2x)^n$ as $2^n x^n$.) Find the interval of convergence of the series, and find an explicit formula for $f(x)$.

Neither the ratio test nor the root test gives the radius of convergence. Nevertheless, that radius can be determined by arguments closely resembling those in the proofs of those tests; cf. "§11.6, Exercise 46".

(Compare Exercise 37 in Stewart. In that exercise, the ratio test does not work, but the root test does.)

§11.8, "Exercise 47". Find the interval of convergence of the power series

$$x + x^2/2 - x^3/3 - x^4/4 + x^5/5 + x^6/6 - x^7/7 - x^8/8 + \ldots .$$

Here the coefficients are the terms of the harmonic series, modified by alternately putting plus signs on two successive terms, then minus signs on the next two. Precisely, the coefficient of x^n is $1/n$ if n has the form $4m+1$ or $4m+2$, and $-1/n$ if n has the form $4m+3$ or $4m+4$.

The radius of convergence should be easy to find; the behavior at the endpoints will require some thought.

§11.8, "Exercise 48". Find the radii of convergence of the following power series:

(a) $\Sigma_0^\infty\, (n^n/n!)\, x^n$ (b) $\Sigma_0^\infty\, (n^n/(2n)!)\, x^n$ (c) $\Sigma_0^\infty\, (n^{2n}/n!)\, x^n$ (d) $\Sigma_0^\infty\, x^{n^2}/n!$

§11.8, "Exercise 49". For any power series $\Sigma_0^\infty a_n x^n$, show that its radius of convergence can be expressed by the formula

$$R = 1/\lim_{N\to\infty} (\text{l.u.b.}(\{\sqrt[n]{|a_n|} \mid n > N\})).$$

Here the l.u.b.'s will be real numbers if the sets over which they are taken are bounded above. We make the convention that the l.u.b. of a set that is not bounded above is the symbol ∞. In evaluating R, we also make the conventions $1/0 = \infty$, $1/\infty = 0$.

These conventions allow the above concise statement of what the radius is in the three situations you will prove can occur. Separating them out, what you will find is the following. First, if for some N the set shown is not bounded above, then it is unbounded for all N, and the radius of convergence of the series is 0. Second, if for all N the set shown is bounded above, and the limit of the least upper bounds is zero, then the radius of convergence is infinite. Finally, if they are bounded above for all N and their least upper bounds do not approach zero, then those bounds approach a finite positive limit, and the reciprocal of that limit, a positive real number, is the radius of convergence of the power series.

§11.9, "Exercise 43". In "§11.8, Exercise 47" I asked you to find the interval of convergence of the power series

$$x + x^2/2 - x^3/3 - x^4/4 + x^5/5 + x^6/6 - x^7/7 - x^8/8 + \ldots,$$

where the coefficient of x^n is $1/n$ if n has the form $4m+1$ or $4m+2$, and $-1/n$ if n has the form $4m+3$ or $4m+4$. Now find a formula for the sum of the above series.

§11.10, "Exercise 87". In this exercise, we obtain, by a third method, the formula $1 - 1/2 + 1/3 - \ldots = \ln 2$ that Stewart develops by two different methods in §11.5, Exercise 36, and Problems Plus 20 on p. 789. Note that this is the formula we would get if we could substitute $x = 1$ in the power series expansion $\ln(1+x) = x - x^2/2 + \ldots + (-1)^{n+1}x^n/n + \ldots$ (cf. Example 6, p. 755). We can't do this so far, because we only have that expansion for $|x| < R = 1$; but the argument below will show how to carry that formula from that case to the case $x = 1$. We start with a general result.

In (a) and (b) below, suppose $f_0, f_1, f_2, \ldots$ are increasing continuous functions (i.e., functions satisfying $f(x_0) \leq f(x_1)$ whenever $x_0 \leq x_1$) on a closed interval $[a, b]$, such that for every $x \in [a, b]$, we have $f_0(x) < f_1(x) < f_2(x) < \ldots$, and such that $\lim_{n\to\infty} f_n(b)$ exists.

(a) Deduce that for every $x \in [a, b]$, $\lim_{n\to\infty} f_n(x)$ exists. (This is a quick application of a fact we have learned.)

(b) Writing $f(x) = \lim_{n\to\infty} f_n(x)$, show that f is continuous at the point b.

(Hint: Given ε, start by showing that we can find an N which puts $f_N(b)$ within $\varepsilon/2$ of $f(b)$, and a δ which puts $f_N(b-\delta)$ within $\varepsilon/2$ of $f_N(b-\delta)$.)

(c) Consider the infinite series $\Sigma_{i=1}^{\infty} (-1)^{i+1}x^i/i$, and for each nonnegative integer n, let $f_n(x)$ be the partial sum $\Sigma_{i=1}^{2n} (-1)^{i+1}x^i/i$ (note the "$2n$" in the range of summation!) Show that the conditions assumed in (a) and (b) above (stated in the sentence before (a)) hold for this sequence of functions.

(d) You already know how to sum the infinite series of part (c) when $|x| < 1$. Use the result of (b) (and some reasoning to take into account the odd partial sums) to determine the sum when $x = 1$.

The remaining two parts are further observations on the general result of (b).

(e) Deduce from (b) that under the assumptions given there, for every $c \in (a, b]$ one has $\lim_{x\to c^-} f(x) = f(c)$.

On the other hand

(f) Show by example that for $c \in [a, b)$, $\lim_{x\to c^+} f(x)$ need not equal $f(c)$. Hence in the context of (b), the only point where we can be sure f is continuous is b. (But of course, when the f_n are partial sums of a power series, as in our application of the above results in parts (c)-(d), then we know that f is continuous everywhere in the interior of its interval of convergence.)

Remark: It is in fact known that *any* power series that converges at one end of its interval of convergence is continuous at that point. But proving that general result requires different methods, which we will not give here.

§11.10, "Exercise 88". (a) Suppose $f(x)$ is an *even* function, given on an interval $(-R, R)$ by a power series in x, where R is a positive real number or ∞. Show that there exists a function $g(x)$ given on $(-R^2, R^2)$ by a power series in x, such that for $x \in [0, R^2)$, one has $g(x) = f(\sqrt{x})$. (Note that this says nothing about the values of $g(x)$ for negative x.) Show also that if R is the radius of convergence of the power series for $f(x)$, then the radius of convergence of the power series for $g(x)$ is R^2.

(b) If for $f(x)$ in part (a) we take the function $\cos x$, show that for negative x, we have $g(x) = \cosh\sqrt{-x}$. Conclude that the function on R which for nonnegative x is given by $\cos\sqrt{x}$, and for negative x by $\cosh\sqrt{-x}$, is infinitely differentiable.

The above equation $g(x) = f(\sqrt{x})$ can also be written $g(x^2) = f(x)$. For variety, I will pose the last part of this problem in the latter form.

(c) Suppose we want a function g given by a power series about 0 which for all nonzero x satisfies $g(x^2) = (\sin x)/x$. Show that such a function exists, again given by a power series that converges for all real x. Determine the values $f(x)$ takes for $x = 0$ and for negative x.

§11.10, "Exercise 89". (This is simply a variant presentation of part (b) of the preceding exercise; so it would not make sense to assign both exercises.)

(a) Show that neither $\cos\sqrt{|x|}$ nor $\cosh\sqrt{|x|}$ is differentiable at $x = 0$. Deduce that neither of them is representable by a power series in any interval containing 0. (Suggestion: shows that each is given by different power series on the two sides of 0.)

(b) On the other hand, show that the function f defined by

$$f(x) = \begin{cases} \cos\sqrt{|x|} & \text{if } x \geq 0 \\ \cosh\sqrt{|x|} & \text{if } x < 0 \end{cases}$$

is given by a power series on the whole real line.

§12.4, "Exercise 55". From the identities Stewart gives you for dot products and cross products (p. 807, boxed list 2, and p. 819, boxed list 11), derive a formula relating $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ and $\mathbf{b}\cdot(\mathbf{a}\times\mathbf{c})$ for all vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

§12.5, "Exercise 84". Let A and B be parallel planes. Show that if L_1 is a line in A and L_2 is a line in B, and L_1 and L_2 are skew, then the distance between L_1 and L_2 is equal to the distance between the planes A and B.

§12.6, "Exercise 54". The equations $z = 1$, $x^2 - y^2 = 1$ define a hyperbola in the plane $z = 1$. Let us call it H.

For each point $P = (x, y, 1)$ of H, let us draw the line through P and the origin, namely $\{(tx, ty, t) \mid t \in \mathbb{R}\}$, and let us call the surface formed by all these lines S. Thus,

$$S = \{(tx, ty, t) \mid (x, y, 1) \in H,\ t \in \mathbb{R}\}.$$

Show that all points of S lie on a cone. What points of that cone, if any, do not lie on S?

(Note: This cone will not be given by an equation of the precise form shown on p. 837, because its axis will not be the z-axis.)

§12.6, "Exercise 55". In the preceding exercise, replace the hyperbola H with a parabola P, also in the plane $z = 1$, and compare the nature of the surface you get with that of the preceding exercise.

§12.6, "Exercise 56". Suppose L_1 and L_2 are skew lines, and c is a positive real number. Show that the locus of points P such that (distance from P to L_1) $= c$ (distance from P to L_2) is a quadric surface. Which type of quadric surface will it be? (The answer will depend on c.)

§12.6, "Exercise 57". This exercise combines some concepts of this course with a fact taught in Math 54, so it is aimed at students in this course who have either taken Math 54, or had a previous algebra course which discussed the existence of nontrivial solutions to systems of m linear equations in n unknowns when $m < n$.

For n a positive integer, let us understand a ''parametric space curve of degree $\leq n$'' to mean any parametric curve of the form

$$\begin{aligned} x &= a_n t^n + \dots + a_1 t + a_0, \\ y &= b_n t^n + \dots + b_1 t + b_0, \\ z &= c_n t^n + \dots + c_1 t + c_0, \end{aligned}$$

where the a's, b's, and c's are real constants with a_n, b_n, c_n not all zero. So, for instance, a parametric space curve of degree 1 is a straight line.

(a) Show that every parametric space curve of degree ≤ 2 lies in a plane; in fact, that if it is not a straight line or a point, then it is a parabola in some plane.

(b) Show that every parametric space curve of degree ≤ 3 lies in a quadric surface; in fact, lies in more than one quadric surface.

(c) Does every parametric space curve of degree ≤ 4 lie in a quadric surface?

(d) Show that not every parametric space curve of degree ≤ 5 lies in a quadric surface. (Suggestion: verify that the curve $x = t^2$, $y = t^3 + t$, $z = t^5$ does not.)

Hint for parts (a)-(c): To show that a given space curve $x = f(t), y = g(t), z = g(t)$ lies in, say, a quadric

$$Ax^2 + Bxy + Cxz + Dy^2 + Eyz + Fz^2 + Gx + Hy + Iz + J = 0$$

is equivalent to showing that there exist real numbers $A, \dots, J$, not all zero, such that when you substitute $f(t)$ for x, $g(t)$ for y and $h(t)$ for z in the above equation, the equation holds for all t. If $f(t)$ etc. are polynomials, the equation will hold for all t if and only if, after substituting and collecting like terms, all terms are zero. Regard this condition as a system of homogeneous linear equations in $A, \dots, J$.

(Stewart gives a version of part (a) above as Problems Plus number 9 to Chapter 13, p. 885.)

§13.1, "Exercise 55". Suppose we are given a space curve

$$x = f(t), \qquad y = g(t), \qquad z = h(t), \qquad a \leq t \leq b,$$

where f, g, h are continuously differentiable functions, and we rotate it about the x-axis to get a surface of revolution. (If you aren't sure what this means, imagine that the curve is embedded as a wiggly wire inside a cube of plastic, and the x-axis is a rod running through the cube, and that we spin the cube about that rod, so that the wire representing our function blurs into a surface.)

Derive a formula for the area of this surface. (Hint: Is this surface the same as the surface of revolution of some curve in the x-y-plane?)

To check your answer, note that if h is the zero function your formula should agree with formula 5 on p. 553 of Stewart, while when $x = 0$, $y = \cos t$, $z = \sin t$, your formula should give 0.

§13.1, "Exercise 56". Suppose we are given a space curve

$$x = f(t), \qquad y = g(t), \qquad z = h(t), \qquad a \le t \le b,$$

where $f,\ g,\ h$ are continuously differentiable functions and $h(t) \ge 0$. Let us define the "curtain" below this curve to consist of all points $(f(t), g(t), z)$ such that $0 \le z \le h(t)$. Derive a formula for the area of this curtain.

Note that if g is the zero function, this should just give the area under the parametrized curve $x = f(t)$, $z = h(t)$ in the x-z-plane. What should it give if h is the constant function 1?

§14.3, "Exercise 96, parts (b)-(d)". Let us call the version of Exercise 96 in Stewart "part (a)". As in that exercise, let a, b and c be the sides of a triangle, and A, B, C the opposite angles.

(b) Taking a, b and c as the independent variables, find $\partial A/\partial a$, $\partial A/\partial b$ and $\partial A/\partial c$.

(c) Taking a, B and C as the independent variables, find $\partial A/\partial a$, $\partial A/\partial B$ and $\partial A/\partial C$.

(The easiest way to do part (b) is, as Stewart suggests for part (a), to write down the Law of Cosines $a^2 = b^2 - 2bc\cos A + c^2$, and apply implicit differentiation. To get part (c), on the other hand, use the formula relating the three angles of any triangle.)

(d) Are the values you get for $\partial A/\partial a$ in parts (b) and (c) the same? Should they be?

Further remarks: For a triangle regarded as determined by the lengths of its three sides, part (b) above asks for essentially all the nontrivial calculations of partial derivatives of sides and angles, since the partials of B and C with respect to a, b and c will be given by formulas like those giving the partials of A, but with the roles of a, b and c permuted; while the partials of a, b and c with respect to each other will be 1 or 0. (Partial with respect to self = 1, partial with respect to any other = 0.)

On the other hand, for a triangle regarded as determined by "angle-side-angle" as in part (c), there are three more nontrivial calculations: the partials of the lengths of one of the other two sides (b or c) with respect to a, B and C. A triangle given by "side-angle-side" yields yet another four partial derivatives to calculate. I didn't assign these additional cases so as not to overload the exercise, but if you're interested, you could work some of them out.

§14.3, "Exercise 106". (a) Suppose g and h are differentiable functions of one variable, and we define a two-variable function by $f(x,y) = g(x) + h(y)$. Show that $f_{xy} = 0$.

The remaining parts of this exercise show you how to prove a harder result, the converse: that any differentiable function f of two variables which satisfies $f_{xy} = 0$ can be written $g(x) + h(y)$ for some functions g and h. Actually, for this to be true we need restrictions on the shape of the domain of f. To keep this exercise simple, we shall assume f is defined on a rectangle. So −

In parts (b)-(d) *below, f will be a differentiable function defined on a rectangle $\{(x,y) \mid a < x < b,\ c < y < d\}$, and satisfying $f_{xy}(x,y) = 0$ for all (x,y) in this rectangle.*

The idea will be to take $g(x)$ to be a function which changes as f does as one moves along some horizontal line, and $h(y)$ to be a function which changes as f does along some vertical line, and to adjust the value of g or h by an additive constant so that where this vertical and horizontal line meet, $g(x) + h(y)$ has the same value as f; and then to prove that $f(x,y) - g(x) - h(y)$ is the zero function, which is equivalent to the desired result that $f(x,y) = g(x) + h(y)$ everywhere.

So let us now take any (x_0, y_0) satisfying $a < x_0 < b, c < y_0 < d$, and define

$$g(x) = f(x, y_0),$$
$$h(y) = f(x_0, y) - f(x_0, y_0),$$
$$e(x, y) = f(x, y) - g(x) - h(y).$$

(b) Show that the function $e(x, y)$ defined above satisfies

$$e(x_0, y) \;=\; 0 \;\text{ for all }\; y \;\text{ such that }\; c < y < d,$$
$$e(x, y_0) \;=\; 0 \;\text{ for all }\; x \;\text{ such that }\; a < x < b.$$
$$e_{xy}(x, y) \;=\; 0 \;\text{ for all }\; (x, y) \;\text{ such that }\; a < x < b \;\text{ and }\; c < y < d.$$

(c) Show that if e is any differentiable function on $\{(x,y) \mid a < x < b,\ c < y < d\}$ which satisfies the three conditions proved in part (b), then its partial derivative e_x is zero at all points of the horizontal line $y = y_0$, and is constant on all vertical lines; hence is everywhere zero. Deduce that e is constant on all horizontal lines, hence is everywhere 0.

(d) Deduce that $f(x, y) = g(x) + h(y)$.

(e) To see that this result is not true for functions on arbitrary regions, let D be the region gotten by removing from the rectangle $\{(x,y) \mid -2 < x < 2,\ -1 < y < 1\}$ the line segment $\{(x,y) \mid -1 \le x \le 1,\ y = 0\}$. Define $f(x, y)$ to be the function on D which is zero everywhere except on the rectangle $\{(x,y) \mid -1 \le x \le 1,\ 0 < y\}$, and is given on that set by $f(x, y) = (1 - x^2)^2$. Show that f is differentiable, and satisfies $f_{xy}(x,y) = 0$ for all (x, y) in D, but that there do not exist a function $g(x)$ on $(-2, 2)$, and a function $h(y)$ on $(-1, 1)$ such that $f(x, y) = g(x) + h(y)$ everywhere on D.

§14.3, "Exercise 107". Part (a) below is background, indicating the first step in a pattern which (b) and (c) continue.

(a) Suppose $a(x)$ is a continuous function of one real variable, and b is a real number. Show that there is a unique differentiable function $f(x)$ such that $f'(x) = a(x)$ for all x, and $f(0) = b$.

(Hint: Use the Fundamental Theorem of Calculus.)

(b) Suppose $a(x, y)$ is a continuous function of two real variables, $b(y)$ is a continuous function of one real variable, and c is a real number. Show that there is a unique function $f(x, y)$ such that for all x and y, $f_x(x, y) = a(x, y)$ (meaning that the partial derivative with respect to x exists, and has the stated value), for all y, $f_y(0, y) = b(y)$ (in the same sense), and $f(0, 0) = c$.

(Suggestion: The last equation determines f at the origin; use this and the next-to-last equation to determine it at all points of the y-axis, and then, given its value at each point of the y-axis, use the first equation to determine f on the horizontal line through that point.)

Using the same method in three dimensions, you should likewise be able to do

(c) Suppose $a(x, y, z)$ is a continuous function of three real variables, $b(y, z)$ is a continuous function of two real variables, $c(z)$ is a continuous function of one real variable, and d is a real number. Show that there is a unique function $f(x, y, z)$ such that $f_x(x, y, z) = a(x, y, z)$ for all x, y and z, $f_y(0, y, z) = b(y, z)$ for all y and z, $f_z(0, 0, z) = c(z)$ for all z, and $f(0, 0, 0) = d$.

§14.5, "Exercise 60". This exercise concerns notation only, not computation or proof. Turn to the displayed formulas in Example 4, p. 940.

(a) Rewrite these formulas using the symbols w_x, x_u etc. for partial derivatives.

(b) The statement of Example 4 involves a function f of four variables, and four functions x, y, z and t of two variables. Hence, rewrite the same formulas using the symbols $f_1, \ldots, f_4$ for the partial derivatives of the function f with respect to its four arguments (i.e., input-positions), x_1 and x_2 for the partial derivatives of the function x with respect to its two arguments and the analogous symbols for the partials of y, z and t with respect to their two arguments, and finally, w_1 and w_2 for the partial derivatives of w with respect to u and v.

(c) Suppose we rename x, y, z, t as x_1, x_2, x_3, x_4 − not meaning the partial derivatives of a function x, but simply four different functions distinguished by subscripts − and similarly rename the independent variables u and v as u_1 and u_2. Moreover, let us denote the partial derivatives of x_1 with respect to u_1 and u_2 by x_{11} and x_{12}, and in general denote the partial derivatives of each x_i with respect to u_1 and u_2 as x_{i1} and x_{i2} respectively. As in part (b), let us write the partials of f with respect to its four arguments as $f_1, \ldots, f_4$, and the partials of w with respect to u_1 and u_2 as w_1, w_2. Using this notation, again rewrite the formulas of Example 4.

(The two formulas you will get can be summarized nicely as a single formula using Σ-notation, but I won't ask you to do this since we haven't reviewed that notation in this course.)

§14.6, "Exercise 71". Show that if u is a differentiable function of 2 or 3 variables, and f a differentiable function of one variable, then $\nabla f(u) = f'(u)\,\nabla u$. Suggestion: use the "general form" of the Chain Rule (p. 940), with $n = 1$, but with the variables named differently.

§15.3, "Exercise 42". Suppose f is a continuous positive real-valued function on an interval $[\alpha, \beta]$ where $0 \le \beta - \alpha < 2\pi$. Obtain by the method of this section a formula for the area of the region defined in polar coordinates by the inequalities $\alpha \le \theta \le \beta$, $0 \le r \le f(\theta)$. Compare your answer with the formulas given on p. 670.

§15.4, "Exercise 34". Consider a lamina with density function $\rho(x, y)$ which occupies a region D. Let us define its *moment* about a vertical line $x = a$, which we will write $M_{(x=a)}$, and its *moment of inertia* about the same line, which we will write $I_{(x=a)}$, to be the values of the double integrals

$$M_{(x=a)} = \iint_D (x-a)\,\rho(x,y)\,dA \qquad \text{and} \qquad I_{(x=a)} = \iint_D (x-a)^2\,\rho(x,y)\,dA.$$

(Thus, for the special case $a = 0$, we have $M_{(x=0)} = M_y$ and $I_{(x=0)} = I_y$.)

(a) Show that for all a, $M_{(x=a)} = M_y - a\,m$, where m is the mass of the lamina. What facts about double integrals do you use in proving this?

(b) Deduce that $M_{(x=a)} = 0$ if and only if the line $x = a$ passes through the center of mass of the lamina.

(c) Obtain a formula for $I_{(x=a)}$ in terms of I_y, M_y, m, and a.

(d) Will there, in general, be a value of a which makes $I_{(x=a)}$ zero? Determine the value of a which makes $I_{(x=a)}$ a minimum.

§16.3, "Exercise 37". Give an "almost-proof" of Theorem 6 on p. 1091 in the case where the domain of $\mathbf{F}$ is a rectangle $\{(x, y) \mid a < x < b,\ c < y < d\}$, as follows: Imitating the method of Example 4 on that page, find a function f which you hope will have gradient $\mathbf{F}$. Show that your method of construction implies that some of the desired equations hold; then use Clairaut's theorem to prove the remaining conditions, *assuming* that the hypothesis of Clairaut's Theorem on continuous second partials is satisfied.

(It is because of this assumption, which I don't ask you to prove, that I have called what you are to give an ''almost proof''. However, in many special cases, this assumption is easy to check; e.g., if P and Q are polynomial functions, in which case the functions you construct will also be polynomials, and so have continuous derivatives of all orders.)

§16.4, ''Exercise 32''. Exercises 22 and 25 of this section (p. 1102) have you prove formulas expressing $\bar{y}$, the y-coordinate of the center of mass of a lamina of constant density, and I_x, the moment of inertia of that lamina with respect to the x-axis, as line integrals around the boundary of the lamina of an integrand only involving dx; and likewise have you express $\bar{x}$ and I_y as line integrals only involving dy.

One can also express $\bar{y}$ and I_x by line integrals only involving dy, and $\bar{x}$ and I_y by line integrals only involving dx. Find and prove such formulas. (The function integrated in each formula can depend on both x and y.)

§16.5, ''Exercise 40''. Give an ''almost proof'' of Theorem 4, p. 1105 by a method analogous to that suggested in ''§16.3, Exercise 37'' above. (Cf. Example 5, p. 1092.)

§16.5, ''Exercise 41''. Suppose $\mathbf{G} = \langle U, V, W \rangle$ is a vector field on $\mathbb{R}^3$ whose component functions have continuous partial derivatives, and which satisfies $\nabla \cdot \mathbf{F} = 0$. Give an ''almost proof'' (as in ''§16.3, Exercise 37'' and ''§16.5, Exercise 40'') that there exists a unique vector field $\mathbf{F} = \langle P, Q, R \rangle$ such that P is everywhere zero, Q is zero on the y-z-plane (i.e., $Q(0, y, z) = 0$ for all y and z), and R zero on the z-axis, and which satisfies $\nabla \times \mathbf{F} = \mathbf{G}$.

(The idea is like that of the two exercises mentioned, but with the complication that for a given $\mathbf{G}$, the $\mathbf{F}$ such that $\nabla \times \mathbf{F} = \mathbf{G}$ is far from unique: If one vector field $\mathbf{F}$ satisfies that equation, then so will $\mathbf{F} + \nabla f$ for every continuously differentiable function f. Hence I have imposed extra conditions to get uniqueness; these will actually make it easier for you to find the desired $\mathbf{F}$. Why the conditions I state are appropriate ones to eliminate the nonuniqueness arising from adding a gradient vector field is suggested by ''§14.3, Exercise 107(c)'' above, which shows how much freedom we have to prescribe the gradient of a function of three variables; but you do not need to do that exercise to do this one.)

§16.6, ''Exercise 65''. Let S be a surface given in spherical coordinates by an equation $\rho = f(\theta, \varphi)$, where f is a positive-valued continuously differentiable function on some region D in $\{(\theta, \varphi) \mid 0 \le \theta \le 2\pi,\ 0 \le \varphi \le \pi\}$. Find a formula for the area of S in terms of f. Suggestion: Try direct calculation from Definition 6 on p.1117 for 10 minutes or until you decide it's too messy, then try the method suggested below.

Suggested method: Let $\mathbf{u}(\theta, \varphi)$ (''$\mathbf{u}$'' standing for ''unit vector'') denote the point of $\mathbb{R}^3$ with spherical coordinates $(1, \theta, \varphi)$. Thus, our surface S is described by the parametric vector equation

$$\mathbf{r}(\theta, \varphi) \ = \ f(\theta, \varphi)\, \mathbf{u}(\theta, \varphi).$$

(a) For $\mathbf{r}(\theta, \varphi)$ given by the above equation, express $\mathbf{r}_\theta$ and $\mathbf{r}_\varphi$ in terms of f and $\mathbf{u}$ and their partial derivatives. (Cf. Theorem 3, p. 858. Don't actually compute the partial derivatives of $\mathbf{u}$ at this stage; that comes next.)

(b) Write out the cartesian coordinates of the vector function $\mathbf{u} = \mathbf{u}(\theta, \varphi)$, and compute its partial derivatives $\mathbf{u}_\theta$ and $\mathbf{u}_\varphi$.

(c) Verify that $\mathbf{u}$, $\mathbf{u}_\theta$ and $\mathbf{u}_\varphi$ are mutually perpendicular. Determine their magnitudes as functions of θ and φ.

(d) Combining the results of (a) and (c), compute $|\mathbf{r}_\theta \times \mathbf{r}_\varphi|$.

(e) Substitute the result from (d) into the Definition on p. 1117 to obtain a formula for the area of S.

§17.1, "Exercise 36". On p. 1156, Stewart leads us through a discovery of the general solution to the linear differential equation with constant coefficients $ay'' + by' + cy = 0$ when the auxiliary equation $ar^2 + br + c = 0$ has distinct real roots, and on p. 1157, he shows us that this solution can be adapted to the case where it has distinct complex conjugate roots. In the case where the equation has two equal roots (i.e., where $b^2 - 4ac = 0$), that approach leads to only one solution $y = e^r$, where r is this root. In that case, Stewart does not show how one might discover a solution linearly independent of this one, but simply hands us such a solution on a platter (p. 1156, bottom and 1157, top).

In class, I showed one way of discovering this solution. This exercise shows you another. The idea is that a quadratic equation with equal roots, r and r, can be written as a *limit* of quadratic equations with distinct roots, r and $r+\varepsilon$, as $\varepsilon \to 0$. Thus, one can try to solve a differential equation whose auxiliary equation has equal roots by taking limits of solutions to differential equations whose auxiliary equations have distinct roots. Let us begin with

(a) If r and ε are real numbers, determine the real numbers $b_{r,r+\varepsilon}$ and $c_{r,r+\varepsilon}$ such that the quadratic equation

$$t^2 + b_{r,r+\varepsilon} t + c_{r,r+\varepsilon} = 0$$

has r and $r+\varepsilon$ as its roots. (Hint: if the polynomial has these two roots, how must it factor?)

(b) For the $b_{r,r+\varepsilon}$ and $c_{r,r+\varepsilon}$ computed above, find the general solution to the differential equation

$$y'' + b_{r,r+\varepsilon} y' + c_{r,r+\varepsilon} y = 0,$$

assuming $\varepsilon \neq 0$. (Hint: you already know the roots of its characteristic equation.)

(c) Find the solution to the above differential equation that satisfies the initial conditions $y(0) = 0$, $y'(0) = 1$. We shall call this function $u_{r,r+\varepsilon}$.

(d) Holding r fixed and letting ε go to zero, compute $\lim_{\varepsilon \to 0} u_{r,r+\varepsilon}(x)$. Call this function $u_{r,r}(x)$.

(e) Setting $\varepsilon = 0$, write down $b_{r,r}$ and $c_{r,r}$, and test whether $u_{r,r}$ satisfies the differential equation

$$y'' + b_{r,r} y' + c_{r,r} y = 0.$$

(f) Verify that $u_{r,r}$ and e^{rx} are linearly independent, and combine them to obtain the general solution to the differential equation of part (e).

Remark: The steps above were all obvious things to do except for (c). If instead of (c) we had just looked at the two exponential solutions and taken their limits as $\varepsilon \to 0$, these limits would have been the same, and not given two linearly independent solutions. But by looking for a solution with properties that cannot be satisfied by an exponential function, and which are preserved on taking limits, one gets a non-exponential solution of the limiting differential equation.

§17.1, "Exercise 37". Determine for what values of a, b, c and d the differential equation $y'' - 6y' + 25y = 0$ with boundary-value conditions $y(a) = b$, $y(c) = d$ has a unique solution, for what values it has no solution, and for what values it has more than one solution. Show that if it has more than one solution, it has infinitely many.

§17.1, "Exercise 38". Generalizing the last part of the above problem, show that if a differential equation $P(x)y'' + Q(x)y' + R(x)y = G(x)$ has more than one solution satisfying some boundary-value conditions

$y(a) = b$, $y(c) = d$, then it has infinitely many solutions satisfying those conditions.

§17.2, "Exercise 29". Suppose $y_p(x)$ is a solution to a differential equation

$$ay'' + by' + cy \;=\; G(x),$$

and suppose that, as in the method of variation of parameters, we have obtained it as a sum

(1) $$y_p \;=\; u_1 y_1 + u_2 y_2$$

where y_1 and y_2 are linearly independent solutions to the homogeneous equation

(2) $$ay'' + by' + cy \;=\; 0,$$

and u_1, u_2 are functions satisfying

(3) $$u_1' y_1 + u_2' y_2 \;=\; 0.$$

Now suppose that for some real number x_0 we find the unique constants c_1 and c_2 such that the function

(4) $$y \;=\; c_1 y_1 + c_2 y_2$$

is tangent to y_p at $x = x_0$; i.e., satisfies the initial conditions

(5) $$y(x_0) \;=\; y_p(x_0), \qquad y'(x_0) \;=\; y_p'(x_0).$$

Prove that these constants will be given by

(6) $$c_1 = u_1(x_0), \qquad c_2 = u_2(x_0).$$

§17.3, "Exercise 19". Suppose $my'' + cy' + ky = 0$ is the equation of an *overdamped* oscillator.

(a) Stewart writes on p. 1170 that "It's possible for the mass to pass through the equilibrium position once, but only once". In particular, he is asserting that no nonzero solution to the above differential equation has more than one zero. Prove that assertion.

(b) Suppose that for each real number a we consider the solution to the above differential equation which satisfies the initial conditions $y(0) = 1$ and $y'(0) = a$. For what values of a will this solution have no zeros? Have one zero with x positive? Have one zero with x negative?

(c) Instead of restricting attention to solutions with $y(0) = 1$, and looking at them in terms of the value of $y'(0)$, let us take the general solution $y = c_1 e^{r_1} + c_2 e^{r_2}$, and require only that c_1 and c_2 not both be zero. Then there are more cases to consider:

Find necessary and sufficient conditions on c_1 and c_2 for y to be everywhere positive, for y to be everywhere negative, for y to have a zero to the left of which it is positive and to the right of which it is negative, and for y to have a zero to the left of which it is negative and to the right of which it is positive. In the latter two cases, determine under what conditions the zero will occur to the left of the origin, at the origin, or to the right of the origin.

§17.3, "Exercise 20". Suppose $my'' + cy' + ky = 0$ is the equation of a *critically damped* oscillator. Examine for this equation the questions analogous to those of the preceding exercise.

§17.4, "Exercise 13". (a) Suppose y is a function defined on the whole real line, which satisfies the differential equation $y''' = y$. Show that the Maclaurin series for y converges to y for all x.

(Hint: Given any x, let M_0 be the maximum value of $|y|$ on the closed interval between 0 and x, let M_1 be the maximum value of $|y'|$ on that interval, and let M_2 be the maximum value of $|y''|$ on that interval.

Taking M to be the largest of these three numbers, show that *all* derivatives of y have absolute values bounded by M on that interval. Then apply the remainder estimate for Taylor series.)

(b) Suppose specifically that $f_0(x)$ is a function satisfying the above differential equation and the initial conditions $f_0(0) = 1$, $f_0'(0) = 0$, $f_0''(0) = 0$, and let us define $f_1(x) = f_0'(x)$, and $f_2(x) = f_0''(x)$. Show that each of these functions also satisfies the given differential equation, say what initial conditions each of them satisfies, and give their Maclaurin series.

(c) For f_0, f_1, f_2 as in part (c), show that *every* solution to the differential equation $y''' = y$ can be written $Af_0(x) + Bf_1(x) + Cf_2(x)$ for some real constants A, B, C. (Hint: Show that the corresponding statement is true for Maclaurin series, and use part (a).)

(d) Show how to express the above function $f_0(x)$ in terms of the functions e^x, $e^{\omega x}$ and $e^{\omega^2 x}$, where ω is the complex cube root of 1 given by

$$\omega = e^{2\pi i/3} = (-1+\sqrt{-3})/2.$$

Deduce corresponding descriptions of $f_1(x)$ and $f_2(x)$.

Remark: The above observations lead to an alternative method of solving Problem Plus number 25 on p. 784 (which was an ''interesting/challenging'' problem note in connection with §11.9). But for that Problem Plus, the method I suggested in class is much quicker.

§17.4, ''Exercise 14''. Consider the differential equation with boundary conditions

$$y'' - y' - y = 0, \quad y(0) = 0, \; y'(0) = 1.$$

(a) Solve the above system as a power series. (Suggestion: First determine a general formula for $y^{(n)}(0)$.)

(b) Solve the above system using the method of §17.1, and express the solution as a power series.

(c) The two power series that you get must be equal; what conclusion do you get when you equate them?

Remark: The same conclusion is gotten in Problem Plus number 24 on p. 790, using a related but different power series.

Appendix F, ''Exercise 1''. We shall obtain here the result omitted in Stewart's proof (p. A46) of L'Hospital's rule (p. A45), namely the case where f and g approach $\pm\infty$. We begin, for simplicity, with the case of where x approaches a finite value from one side.

(a) Prove that if $\lim_{x\to a^+} |g(x)| = \infty$ and if $\lim_{x\to a^+} f'(x)/g'(x)$ is a real number L, then

$$\lim_{x\to a^+} f(x)/g(x) = L.$$

(Suggested outline: Given ε, first choose δ_1 such that for all $x \in (a, a+\delta_1)$ we have

$$|f'(x)/g'(x) - L| < \min(1, \varepsilon/3).$$

Then show that we can choose $\delta \le \delta_1$ such that for all $x \in (a, a+\delta)$, $g(x)$ is large enough to force the two conditions

$$|f(a+\delta_1)|/|g(x) - g(a+\delta_1)| < \varepsilon/3$$

and

$$(|L| + 2\varepsilon/3)\,|g(a+\delta_1)/g(x)| < \varepsilon/3.$$

Now for any $x \in (a, a+\delta)$ apply Cauchy's Mean Value Theorem to f and g on the interval $[x, a+\delta_1]$, and show that for the resulting c, each of the following four terms differs from the next by $< \varepsilon/3$: (i) L, (ii) the common value of the two sides of the equation given by Cauchy's Mean Value Theorem, (iii) the right-hand side of that equation with the $-f(a+\delta_1)$ removed from the numerator, and (iv) the same expression with the $-g(a+\delta_1)$ also removed from the denominator. Verify that this shows that $|f(x)/g(x) - L| < \varepsilon$.)

(b) Show how to deduce from the above result, first, the corresponding statement for $x \to a^-$, then the statement for $x \to a$; then the corresponding results with $g(x) \to -\infty$, and finally, the corresponding statements where a is infinite, rather than a real number.

Appendix H, "Exercise 51". If a and b are real numbers, and n a nonnegative integer, let $S(a,b,n) = \Sigma_{k=0}^{n} \sin(ak+b)$.

(a) Assuming a is not an exact multiple of 2π, find a formula in "closed form" (i.e., without summation signs) for $S(a,b,n)$. (Suggestion: Use Exercise 48, together with what you know about summing finitely many terms of a geometric series. The proof of the formula for summing a geometric series is based on laws of arithmetic that hold for complex as well as for real numbers, hence it is true for complex numbers as for real numbers.)

(b) Find a formula for $S(a,b,n)$ in the case not covered by (a), namely, when a is an integer multiple of 2π.

(c) Show that in case (a) above, for any values of a and b, the sequence $(S(a,b,n))_{n=0}^{\infty}$ is bounded.

(d) What can be said about the boundedness or unboundedness of $(S(a,b,n))_{n=0}^{\infty}$ in case (b)?

(e) Show that if in case (a) above one lets $U(a,b)$ denote the least upper bound of the sequence $(S(a,b,n))_{n=0}^{\infty}$, then for any b, $\lim_{a\to 0} |U(a,b)| = \infty$. (It is difficult to give an exact formula for $U(a,b)$, but one can show that it becomes large by finding values that it exceeds.)

Appendix H, "Exercise 52". Find a formula for the function $\sin^5 x$ (i.e., $(\sin x)^5$) of the form

$$\sin^5 x \;=\; c_1 \sin x + c_2 \sin 2x + c_3 \sin 3x + \ldots + c_n \sin nx,$$

where n is some positive integer, and $c_1, \ldots, c_n$ are real constants. (Suggestion: make use of the formula $\sin x = (e^{ix} - e^{-ix})/2i$.)